

HEAT KERNEL AND CURVATURE BOUNDS IN RICCI FLOWS WITH BOUNDED SCALAR CURVATURE — PART II

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ABSTRACT. In this paper we analyze the behavior of the distance function under Ricci flows whose scalar curvature is uniformly bounded. We will show that on small time-intervals the distance function is $\frac{1}{2}$ -Hölder continuous in a uniform sense. This implies that the distance function can be extended continuously up to the singular time.

1. INTRODUCTION

In this paper, we extend the estimates of [BZ15], to prove the following result:

Theorem 1.1. *For any $0 < A < \infty$ and $n \in \mathbb{N}$ there is a constant $C = C(A, n) < \infty$ such that the following holds:*

Let $(\mathbf{M}^n, (g_t)_{t \in [0,1]})$ be a Ricci flow ($\partial_t g_t = -2 \operatorname{Ric}_{g_t}$) on an n -dimensional compact manifold \mathbf{M} with the property that $\nu[g_0, 1 + A^{-1}] \geq -A$. Assume that the scalar curvature satisfies $|R| \leq R_0$ on $\mathbf{M} \times [0, 1]$ for some constant $0 \leq R_0 \leq A$.

Then for any $0 \leq t_1 \leq t_2 \leq 1$ and $x, y \in \mathbf{M}$ we have the distance bound

$$d_{t_1}(x, y) - C\sqrt{t_2 - t_1} \leq d_{t_2}(x, y) \leq \exp(CR_0^{1/2}\sqrt{t_2 - t_1})d_{t_1}(x, y) + C\sqrt{t_2 - t_1}.$$

In particular, if $\min\{d_{t_1}(x, y), d_{t_2}(x, y)\} \leq D$ for some $D < \infty$, then

$$|d_{t_1}(x, y) - d_{t_2}(x, y)| \leq C'\sqrt{t_2 - t_1},$$

where C' may depend on A , D and n .

By parabolic rescaling, we obtain distance bounds on larger time-intervals. Note that Theorem 1.1 is a generalization of [BZ15, Theorem 1.1], which only provides a bound on the distance distortion that does not improve for t_2 close to t_1 . The constant $\nu[g_0, 1 + A^{-1}]$ is defined as the infimum of Perelman's μ -functional (cf [Per02]) $\mu[g_0, \tau]$ over all $\tau \in (0, 1 + A^{-1})$. For more details see [BZ15, sec 2]. The condition $\nu[g_0, 1 + A^{-1}] \geq -A$, can be viewed as a non-collapsing condition. The exponential factor in the upper bound is necessary, as one can see for example in the case in which $(\mathbf{M}, (g_t)_{t \in [0,1]})$ is the Ricci flow on a hyperbolic manifold and the distance between x, y is very large. The proof of Theorem 1.1 will heavily use the results of [BZ15], in particular the heat kernel bound, [BZ15, Theorem 1.4].

As a consequence of Theorem 1.1, we obtain the following:

Corollary 1.2. *Let $(\mathbf{M}, (g_t)_{t \in [0, T)})$, $T < \infty$ be a Ricci flow on a compact manifold and assume that the scalar curvature satisfies $R < C < \infty$ on $\mathbf{M} \times [0, T)$. Then the distance function*

$$d : \mathbf{M} \times \mathbf{M} \times [0, T) \longrightarrow [0, \infty), \quad (x, y, t) \longmapsto d_t(x, y)$$

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can be extended continuously onto the domain $\mathbf{M} \times \mathbf{M} \times [0, T]$.

Note that the corollary does not state that $d_T : \mathbf{M} \times \mathbf{M} \rightarrow [0, \infty)$ is a metric on \mathbf{M} . It only follows that d_T is a pseudometric, which means that we may have $d_T(x, y) = 0$ for some $x \neq y$. After taking the metric identification, however, $(\mathbf{M}/\sim, d_T)$ is in fact the Gromov-Hausdorff limit of (\mathbf{M}, g_t) as $t \nearrow T$. Here $x \sim y$ if and only if $d_T(x, y) = 0$. Moreover, since the volume measure converges as well, the space $(\mathbf{M}/\sim, d_T)$ becomes a metric measure space with doubling property and this space is the limit of (\mathbf{M}, g_t) in the measured Gromov-Hausdorff sense.

More generally, we obtain the following consequence of Theorem 1.1.

Corollary 1.3. *Let $(\mathbf{M}^i, (g_t^i)_{t \in [0, 1]})$ be a sequence of Ricci flows on n -dimensional compact manifolds \mathbf{M}^i with the property that $\nu[g_0^i, 1 + A^{-1}] \geq -A$ and $|R| < A$ on $\mathbf{M} \times [0, 1]$ for some uniform $A < \infty$. Let $x_i \in \mathbf{M}^i$ be points. Then, after passing to a subsequence, we can find a pointed metric space $(\overline{\mathbf{M}}, \overline{d}, \overline{x})$, a continuous function*

$$d^\infty : \overline{\mathbf{M}} \times \overline{\mathbf{M}} \times [0, 1] \rightarrow [0, \infty), \quad (x, y, t) \mapsto d_t^\infty(x, y)$$

and a continuous family of measures $(\mu_t)_{t \in [0, 1]}$ such that for any $x, y \in \overline{\mathbf{M}}$, the function $t \mapsto d_t^\infty(x, y)$ is $\frac{1}{2}$ -Hölder continuous and such that for any $t \in [0, 1]$, the metric identification $(\overline{\mathbf{M}}/\sim_t, d_t^\infty, \mu_t, \overline{x})$ is a metric measure space with doubling property for balls of radius less than \sqrt{t} . Here $x \sim_t y$ if and only if $d_t^\infty(x, y) = 0$. Moreover, for any $t \in [0, 1]$ the sequence $(\mathbf{M}^i, g_t^i, dg_t^i, x_i)$ converges to $(\overline{\mathbf{M}}/\sim_t, d_t^\infty, \mu_t, \overline{x})$ in the pointed, measured Gromov-Hausdorff sense.

For the proof of Corollary 1.3 see section 5.

Note that if we impose the extra assumption that $|R| < R_i$ on $\mathbf{M} \times [0, 1]$ for some sequence R_i with $\lim_{i \rightarrow \infty} R_i = 0$, then the limiting family of measures $(\mu_t)_{t \in [0, 1]}$ is constant in time. Unfortunately, however, our results do not imply that $(d_t^\infty)_{t \in [0, 1]}$ is constant in time as well.

Finally, we mention a direct consequence of Theorem 1.1, which can be interpreted as an analogue of the main result of [CN12] in the parabolic case.

Corollary 1.4. *For any $0 < A < \infty$ and $n \in \mathbb{N}$ there is a constant $C = C(A, n) < \infty$ such that the following holds:*

Let $(\mathbf{M}^n, (g_t)_{t \in [0, 1]})$ be a Ricci flow on an n -dimensional compact manifold \mathbf{M} with the property that $\nu[g_0, 1 + A^{-1}] \geq -A$. Assume that the scalar curvature satisfies $|R| \leq A$ on $\mathbf{M} \times [0, 1]$.

Then for any $r > 0$ and $0 \leq t_1 \leq t_2 \leq 1$ and $x \in \mathbf{M}$ we have the following bound for Gromov-Hausdorff distance of r -balls

$$d_{\text{GH}}(B(x, t_1, r), B(x, t_2, r)) \leq C\sqrt{|t_1 - t_2|}.$$

For the rest of the paper, we will fix the dimension $n \geq 2$ of the manifold \mathbf{M} . Most of our constants will depend on n . For convenience we will not mention this dependence anymore.

2. UPPER VOLUME BOUND

We first generalize the upper volume bound from [Zha12] or [CW13].

Lemma 2.1. *For any $A < \infty$ there is a uniform constant $C_0 = C_0(A) < \infty$ such that the following holds:*

Let $(\mathbf{M}^n, (g_t)_{t \in [-1, 1]})$ be a Ricci flow on a compact, n -dimensional manifold \mathbf{M} with $|R| \leq 1$ on $\mathbf{M} \times [-1, 1]$. Assume that $\nu[g_{-1}, 4] \geq -A$. Then for any $(x, t) \in \mathbf{M} \times [0, 1]$ and $r > 0$ we have

$$|B(x, t, r)|_t < C_0 r^n e^{C_0 r}.$$

Here $|S|_t$ denotes the volume of a set $S \subset \mathbf{M}$ with respect to the metric g_t .

Proof. It follows from [Per02], [Zha12], [CW13] (see also [BZ15, sec 2]), that for any $x \in \mathbf{M}$ and $0 \leq r \leq 1$, we have

$$(2.1) \quad cr^n \leq |B(x, t_0, r)|_{t_0} \leq Cr^n,$$

for some constants c, C , which only depend on A .

Fix some $x \in \mathbf{M}$ and let $N < \infty$ be maximal with the property that we can find points $x_1, \dots, x_N \in B(x, t, \frac{1}{2})$ such that the balls $B(x_1, t, \frac{1}{8}), \dots, B(x_N, t, \frac{1}{8})$ are pairwise disjoint. Note that then

$$B(x_1, t, \frac{1}{8}), \dots, B(x_N, t, \frac{1}{8}) \subset B(x, t, 1).$$

So, by (2.1), we have $N \leq C_* := (c(\frac{1}{8})^n)^{-1}C$. Moreover, by the maximality of N , we have

$$(2.2) \quad B(x_1, t, \frac{1}{4}) \cup \dots \cup B(x_N, t, \frac{1}{4}) \supset B(x, t, \frac{1}{2}).$$

We now argue that for all $r \geq \frac{1}{2}$

$$(2.3) \quad B(x_1, t, r) \cup \dots \cup B(x_N, t, r) \supset B(x, t, r + \frac{1}{4}).$$

Let $y \in B(x, t, r + \frac{1}{4})$ and consider a time- t minimizing geodesic $\gamma : [0, l] \rightarrow \mathbf{M}$ between x and y that is parameterized by arclength. Then $l < r + \frac{1}{4}$. By (2.2) we may pick $i \in \{1, \dots, N\}$ such that $\gamma(\frac{1}{2}) \in \overline{B(x_i, t, \frac{1}{4})}$. Then

$$\text{dist}_t(x_i, y) \leq (l - \frac{1}{2}) + \text{dist}_t(\gamma(\frac{1}{2}), x_i) \leq l - \frac{1}{4} < r.$$

So $y \in B(x_i, t_0, r)$, which confirms (2.3).

Let us now prove by induction on $k = 1, 2, \dots$ that for any $x \in \mathbf{M}$

$$(2.4) \quad |B(x, t, \frac{1}{4}k)|_t < C_*^k.$$

For $k = 1$, the inequality follows from (2.1) (assuming $c < 1$ and hence $C_* > C$). If the inequality is true for k , then we can use (2.3) to conclude

$$|B(x, t, \frac{1}{4}(k+1))|_t \leq |B(x_1, t, \frac{1}{4}k)|_t + \dots + |B(x_N, t, \frac{1}{4}k)|_t \leq N \cdot C_*^k \leq C_* \cdot C_*^k = C_*^{k+1}.$$

So (2.4) also holds for $k+1$. This finishes the proof of (2.4).

The assertion of the lemma now follows from (2.1) for $r < 1$. For $r \geq 1$ choose $k \in \mathbb{N}$ such that $\frac{1}{4}(k-1) \leq r < \frac{1}{4}k$. Then, by (2.4), we have

$$|B(x, t, r)|_t < |B(x, t, \frac{1}{4}k)|_t < C_*^k = C_* e^{(\log C_*)(k-1)} \leq C_* e^{4(\log C_*)r}.$$

This finishes the proof. \square

3. GENERALIZED MAXIMUM PRINCIPLE

Consider a Ricci flow $(g_t)_{t \in I}$ on a closed manifold \mathbf{M} . In the following we will consider the heat kernel $K(x, t; y, s)$ on a Ricci flow background. That is, for any $(y, s) \in \mathbf{M} \times I$ the kernel $K(\cdot, \cdot; y, s)$ is defined for $t > s$ and $x \in \mathbf{M}$ and satisfies

$$(\partial_t - \Delta_x)K(x, t; y, s) = 0 \quad \text{and} \quad \lim_{t \searrow s} K(\cdot, t; y, s) = \delta_y.$$

Then, for fixed $(x, t) \in \mathbf{M} \times I$, the function $K(x, t; \cdot, \cdot)$, which is defined for $s < t$, is a kernel for the conjugate heat equation

$$(-\partial_s - \Delta_y + R(y, s))K(x, t; y, s) = 0 \quad \text{and} \quad \lim_{s \nearrow t} K(x, t; \cdot, s) = \delta_x.$$

Recall that for any $s < t$ and $x \in \mathbf{M}$ we have

$$(3.1) \quad \int_{\mathbf{M}} K(x, t; y, s) dg_s(y) = 1.$$

Lemma 3.1. *Let $(\mathbf{M}, (g_t)_{t \in [0,1]})$ be a Ricci flow on a compact manifold \mathbf{M} with $|R| \leq R_0$ on $\mathbf{M} \times [0, 1]$ for some constant $R_0 \geq 0$. Then for any $(x, t) \in \mathbf{M} \times (0, 1]$ we have*

$$\int_0^t \int_{\mathbf{M}} K(x, t; y, s) |\text{Ric}|^2(y, s) dg_s(y) ds \leq R_0.$$

Proof. This follows from the identities

$$R(x, t) = \int_{\mathbf{M}} K(x, t; y, 0) R(y, 0) dg_0(y) + 2 \int_0^t \int_{\mathbf{M}} K(x, t; y, s) |\text{Ric}|^2(y, s) dg_s(y) ds$$

and (3.1) as well as $R(x, t) \leq R_0$ and $R(\cdot, 0) \geq -R_0$ on \mathbf{M} . \square

We will now use the Gaussian bounds from [BZ15] to bound the forward heat kernel in terms of the backwards conjugate heat kernel based at a certain point and time. Note that in the following Lemma we only obtain estimates on the time-interval $[0, 1]$, but we need to assume that the flow exists on $[-1, 1]$. This is due to an extra condition in [BZ15, Theorem 1.4].

Lemma 3.2. *For any $A < \infty$ there are uniform constants $C_1 = C_1(A), Y = Y(A) < \infty$ such that the following holds:*

Let $(\mathbf{M}^n, (g_t)_{t \in [-1,1]})$ be a Ricci flow on a compact, n -dimensional manifold \mathbf{M} with the property that $\nu[g_{-1}, 4] \geq -A$. Assume that $|R| \leq 1$ on $\mathbf{M} \times [-1, 1]$. Let $0 \leq t_1 < t_2 < t_3 \leq 1$ such that

$$Y(t_2 - t_1) \leq t_3 - t_2 \leq 10Y(t_2 - t_1).$$

Then for all $x, y \in \mathbf{M}$

$$K(x, t_2; y, t_1) < C_1 K(y, t_3; x, t_2).$$

Proof. Recall that, by [BZ15, Theorem 1.4], there are constants $C_1^* = C_1^*(A), C_2^* = C_2^*(A) < \infty$ such that for any $0 \leq s < t \leq 1$

$$(3.2) \quad \frac{1}{C_1^*(t-s)^{n/2}} \exp\left(-\frac{C_2^* d_s^2(x, y)}{t-s}\right) < K(x, t; y, s) < \frac{C_1^*}{(t-s)^{n/2}} \exp\left(-\frac{d_s^2(x, y)}{C_2^*(t-s)}\right).$$

Set now

$$Y := (C_2^*)^2 \quad \text{and} \quad C_1 := (C_1^*)^2 (10Y)^{n/2}.$$

Then

$$\begin{aligned}
K(x, t_2; y, t_1) &< \frac{C_1^*}{(t_2 - t_1)^{n/2}} \exp\left(-\frac{d_{t_1}^2(x, y)}{C_2^*(t_2 - t_1)}\right) \\
&\leq \frac{C_1^*}{(10Y)^{-n/2}(t_3 - t_2)^{n/2}} \exp\left(-\frac{d_{t_1}^2(x, y)}{C_2^*(t_2 - t_1)}\right) \\
&\leq C_1 \frac{1}{C_1^*(t_3 - t_2)^{n/2}} \exp\left(-\frac{d_{t_1}^2(x, y)}{C_2^* Y^{-1}(t_3 - t_2)}\right) \\
&= C_1 \frac{1}{C_1^*(t_3 - t_2)^{n/2}} \exp\left(-\frac{C_2^* d_{t_1}^2(x, y)}{(t_3 - t_2)}\right) < C_1 K(y, t_3, x, t_2).
\end{aligned}$$

This finishes the proof. \square

Next, we combine Lemmas 3.1 and 3.2 to obtain the following bound.

Lemma 3.3. *For any $A < \infty$ there are uniform constants $C_2 = C_2(A) < \infty$, $\theta_2 = \theta_2(A) > 0$ such that the following holds:*

Let $(\mathbf{M}^n, (g_t)_{t \in [-1, 1]})$ be a Ricci flow on a compact, n -dimensional manifold \mathbf{M} with the property that $\nu[g_{-1}, 4] \geq -A$. Assume that $|R| \leq R_0$ on $\mathbf{M} \times [-1, 1]$ for some constant $0 \leq R_0 \leq 1$. Then for any $0 \leq t < 1$ and $0 < a \leq \theta_2(1 - t)$ and $x \in \mathbf{M}$ we have

$$\int_{t+a}^{t+2a} \int_{\mathbf{M}} K(y, s; x, t) |\text{Ric}|(y, s) dg_s(y) ds < C_2 R_0^{1/2} \sqrt{a}.$$

Proof. Choose $\theta_2 := \frac{1}{2}Y^{-1}$ and set

$$t_3 := t + 2Ya \leq 1.$$

So for any $s \in [t + a, t + 2a]$ we have

$$Y(s - t) \leq Y \cdot 2a = t_3 - t \leq 10Ya \leq 10Y(s - t).$$

So by Lemma 3.2, we have for any $(y, s) \in \mathbf{M} \times [t + a, t + 2a]$

$$K(y, s; x, t) < C_1 K(x, t_3; y, s).$$

We can then conclude, using Cauchy-Schwarz, (3.1) and Lemma 3.1, that

$$\begin{aligned}
&\int_{t+a}^{t+2a} \int_{\mathbf{M}} K(y, s; x, t) |\text{Ric}|(y, s) dg_s(y) ds \\
&\leq C_1 \int_{t+a}^{t+2a} \int_{\mathbf{M}} K(x, t_3; y, s) |\text{Ric}|(y, s) dg_s(y) ds \\
&\leq C_1 \left(\int_{t+a}^{t+2a} \int_{\mathbf{M}} K(x, t_3; y, s) dg_s(y) ds \right)^{1/2} \\
&\quad \cdot \left(\int_{t+a}^{t+2a} \int_{\mathbf{M}} K(x, t_3; y, s) |\text{Ric}|^2(y, s) dg_s(y) ds \right)^{1/2} \\
&= C_1 \sqrt{a} \left(\int_{t+a}^{t+2a} \int_{\mathbf{M}} K(x, t_3; y, s) |\text{Ric}|^2(y, s) dg_s(y) ds \right)^{1/2} \\
&\leq C_1 R_0^{1/2} \sqrt{a}.
\end{aligned}$$

This proves the desired result. \square

Lemma 3.4. *For any $A < \infty$ there are constants $C_3 = C_3(A) < \infty$, $\theta_3 = \theta_3(A) > 0$ such that the following holds:*

Let $(\mathbf{M}^n, (g_t)_{t \in [0,1]})$ be a Ricci flow on a compact, n -dimensional manifold \mathbf{M} with the property that $\nu[g_{-1}, 4] \geq -A$. Assume that $|R| \leq R_0$ on $\mathbf{M} \times [-1, 1]$ for some constant $0 \leq R_0 \leq 1$. Then for any $0 \leq s < t \leq 1$ with $t - s \leq \theta_3(1 - s)$ and any $x \in \mathbf{M}$, we have

$$\int_s^t \int_{\mathbf{M}} K(y, s; x, t) |\text{Ric}|(y, s) dg_s(y) ds < C_3 R_0^{1/2} \sqrt{t - s}.$$

Proof. Choose $\theta_3(A) = \theta_2(A)$. Then, using Lemma 3.3,

$$\begin{aligned} & \int_s^t \int_{\mathbf{M}} K(y, s; x, t) |\text{Ric}|(y, s) dg_s(y) \\ &= \sum_{k=1}^{\infty} \int_{s+(t-s)2^{-k}}^{s+2(t-s)2^{-k}} \int_{\mathbf{M}} K(y, s; x, t) |\text{Ric}|(y, s) dg_s(y) ds \\ &\leq \sum_{k=1}^{\infty} C_2 R_0^{1/2} \sqrt{(t-s)2^{-k}} \\ &= C_2 R_0^{1/2} \sqrt{t-s} \sum_{k=1}^{\infty} 2^{-k/2} \\ &\leq C C_2 R_0^{1/2} \sqrt{t-s}. \end{aligned}$$

This proves the desired estimate. \square

Proposition 3.5. *For every $A < \infty$ there are constants $\theta_4 = \theta_4(A) > 0$ and $C_4 = C_4(A) < \infty$ such that the following holds:*

Let $(\mathbf{M}^n, (g_t)_{t \in [-1,1]})$ be a Ricci flow on a compact, n -dimensional manifold \mathbf{M} with the property that $\nu[g_{-1}, 4] \geq -A$. Assume that $|R| \leq R_0$ on $\mathbf{M} \times [-1, 1]$ for some constant $0 \leq R_0 \leq 1$. Let $H > 1$ and $[t_1, t_2] \subset [0, 1]$ be a sub-interval with $t_2 - t_1 \leq \theta_4 \min\{(1-t_1), H^{-1}\}$ and consider a non-negative function $f \in C^\infty(\mathbf{M} \times [t_1, t_2])$ that satisfies the following evolution inequality in the barrier sense:

$$-\partial_t f \leq \Delta f + H |\text{Ric}| f - R f.$$

Then

$$\max_{\mathbf{M}} f(\cdot, t_1) \leq (1 + C_4 H R_0^{1/2} \sqrt{t_2 - t_1}) \max_{\mathbf{M}} f(\cdot, t_2).$$

Note that with similar techniques, we can analyze the evolution inequality $-\partial_t f \leq \Delta f + H |\text{Ric}|^p f$ for any $p \in (0, 2)$.

Proof. We first find that for any $(x, t) \in \mathbf{M} \times [-1, 1]$ and $t < s \leq 1$

$$\begin{aligned} \frac{d}{ds} \int_{\mathbf{M}} K(y, s; x, t) dg_s(y) &= \int_{\mathbf{M}} (\Delta_y K(y, s; x, t) - K(y, s; x, t) R(y, s)) dg_s(y) \\ &\leq R_0 \int_{\mathbf{M}} K(y, s; x, t) dg_s(y), \end{aligned}$$

which implies

$$\int_{\mathbf{M}} K(y, s; x, t) dg_s(y) \leq e^{R_0(s-t)}.$$

So for any $(x, t) \in \mathbf{M} \times [t_1, t_2]$ we have by Lemma 3.4, assuming $\theta_4 \leq \theta_3$ and $C_3 > 1$,

$$\begin{aligned} f(x, t) &\leq \int_{\mathbf{M}} K(y, t_2; x, s) f(y, t_2) dg_{t_2}(y) \\ &\quad + \int_t^{t_2} \int_{\mathbf{M}} K(y, s; x, t) \cdot H|\text{Ric}|(y, s) \cdot f(y, s) dg_s(y) ds \\ &\leq e^{R_0(t_2-t)} \max_{\mathbf{M}} f(\cdot, t_2) + H \left(\max_{\mathbf{M} \times [t, t_2]} f \right) \int_t^{t_2} \int_{\mathbf{M}} K(y, s; x, t) |\text{Ric}|(y, s) dg_s(y) ds \\ &\leq e^{R_0(t_2-t)} \max_{\mathbf{M}} f(\cdot, t_2) + H \left(\max_{\mathbf{M} \times [t, t_2]} f \right) \cdot C_3 R_0^{1/2} \sqrt{t_2 - t}. \end{aligned}$$

It follows that

$$\max_{\mathbf{M} \times [t, t_2]} f \leq e^{R_0(t_2-t)} \max_{\mathbf{M}} f(\cdot, t_2) + \left(\max_{\mathbf{M} \times [t, t_2]} f \right) \cdot C_3 H R_0^{1/2} \sqrt{t_2 - t}.$$

So if $t_2 - t < (2C_3H)^{-2}$, then

$$\max_{\mathbf{M} \times [t, t_2]} f \leq \frac{e^{R_0(t_2-t)} \max_{\mathbf{M}} f(\cdot, t_2)}{1 - C_3 H R_0^{1/2} \sqrt{t_2 - t}} \leq (1 + 10C_3 H R_0^{1/2} \sqrt{t_2 - t}) \max_{\mathbf{M}} f(\cdot, t_2).$$

This finishes the proof. \square

4. PROOF OF THEOREM 1.1

We will first establish a lower bound on the distortion of the distance:

Lemma 4.1. *For every $A < \infty$ there is a constant $C_5 = C_5(A) < \infty$ such that the following holds:*

Let $(\mathbf{M}^n, (g_t)_{t \in [-1, 1]})$ be a Ricci flow on a compact, n -dimensional manifold \mathbf{M} with the property that $\nu[g_{-1}, 4] \geq -A$. Assume that $|R| \leq 1$ on $\mathbf{M} \times [-1, 1]$. Let $[t_1, t_2] \subset [0, 1]$ be a sub-interval and consider two points $x_1, x_2 \in \mathbf{M}$. Then

$$d_{t_2}(x_1, x_2) \geq d_{t_1}(x_1, x_2) - C_5 \sqrt{t_2 - t_1}.$$

Proof. Set $d := d_{t_1}(x_1, x_2)$ and let $u \in C^0(\mathbf{M} \times [t_1, t_2]) \cap C^\infty(\mathbf{M} \times (t_1, t_2))$ be a solution to the heat equation

$$\partial_t u = \Delta u, \quad u(\cdot, t_1) = d_{t_1}(x_1, \cdot).$$

Then for any $(x, t) \in \mathbf{M} \times [t_1, t_2]$

$$u(x, t) = \int_{\mathbf{M}} K(x, t; y, t_1) u(t_1) dg_{t_1}(y) = \int_{\mathbf{M}} K(x, t; y, t_1) d_{t_1}(x_1, y) dg_{t_1}(y).$$

Using [BZ15, Theorem 1.4] (compare also with (3.2)), we find that by Lemma 2.1

$$\begin{aligned}
u(x_1, t_2) &\leq \int_{\mathbf{M}} \frac{C_1^*}{(t_2 - t_1)^{n/2}} \exp\left(-\frac{d_{t_1}^2(x_1, y)}{C_2^*(t_2 - t_1)}\right) d_{t_1}(x_1, y) dg_{t_1}(y) \\
&= \sum_{k=-\infty}^{\infty} \int_{B(x_1, t_1, 2^k) \setminus B(x_1, t_1, 2^{k-1})} \frac{C_1^*}{(t_2 - t_1)^{n/2}} \exp\left(-\frac{d_{t_1}^2(x_1, y)}{C_2^*(t_2 - t_1)}\right) \\
&\quad \cdot d_{t_1}(x_1, y) dg_{t_1}(y) \\
&\leq \sum_{k=-\infty}^{\infty} |B(x_1, t_1, 2^k)|_{t_1} \frac{C_1^*}{(t_2 - t_1)^{n/2}} \exp\left(-\frac{2^{2k-2}}{C_2^*(t_2 - t_1)}\right) \cdot 2^k \\
&\leq \sum_{k=-\infty}^{\infty} C_0 (2^k)^n e^{C_0 2^k} \frac{C_1^*}{(t_2 - t_1)^{n/2}} \exp\left(-\frac{2^{2k}}{4C_2^*(t_2 - t_1)}\right) \cdot 2^k \\
&\leq \int_{\mathbb{R}^n} \frac{CC_0 C_1^*}{(t_2 - t_1)^{n/2}} \exp\left(2C_0|x| - \frac{|x|^2}{4C_2^*(t_2 - t_1)}\right) |x| dx \\
&= \sqrt{t_2 - t_1} \int_{\mathbb{R}^n} CC_0 C_1^* \exp\left(2C_0|x|\sqrt{t_2 - t_1} - \frac{|x|^2}{4C_2^*}\right) |x| dx \leq C\sqrt{t_2 - t_1}
\end{aligned}$$

On the other hand, using (3.1),

$$\begin{aligned}
|d - u(x_2, t_2)| &= \left| \int_{\mathbf{M}} K(x_2, t; y, t_1)(d - d_{t_1}(x_1, y)) dg_{t_1}(y) \right| \\
&\leq \int_{\mathbf{M}} K(x_2, t; y, t_1) |d_{t_1}(x_1, x_2) - d_{t_1}(x_1, y)| dg_{t_1}(y) \leq \int_{\mathbf{M}} K(x_2, t; y, t_1) d_{t_1}(x_2, y) dg_{t_1}(y).
\end{aligned}$$

So similarly,

$$|d - u(x_2, t_2)| \leq C\sqrt{t_2 - t_1}.$$

It follows that

$$(4.1) \quad |u(x_1, t_2) - u(x_2, t_2)| \geq d - 2C\sqrt{t_2 - t_1}.$$

Next, consider the quantity $|\nabla u|$ on $\mathbf{M} \times [t_1, t_2]$. It is not hard to check that, in the barrier sense,

$$(4.2) \quad \partial_t |\nabla u| \leq \Delta |\nabla u|.$$

Since $|\nabla u|(\cdot, t_1) \leq 1$, we have by the maximum principle that $|\nabla u| \leq 1$ on $\mathbf{M} \times [t_1, t_2]$. So

$$|u(x_1, t_2) - u(x_2, t_2)| \leq d_{t_2}(x_1, x_2).$$

Together with (4.1) this gives us

$$d_{t_2}(x_1, x_2) \geq d - 2C\sqrt{t_2 - t_1} = d_{t_1}(x_1, x_2) - 2C\sqrt{t_2 - t_1}.$$

This finishes the proof. \square

For the upper bound on the distance distortion, we will argue similarly, by reversing time. The derivation of the bound on $|\nabla u|$ will now be more complicated, since the equation (4.2) will have an extra $4|\text{Ric}| |\nabla u|$ term. We will overcome this difficulty by applying the generalized maximum principle from Proposition 3.5.

Lemma 4.2. *For every $A < \infty$ there are constants $\theta_6 = \theta_6(A) > 0$ and $C_6 = C_6(A) < \infty$ such that the following holds:*

Let $(\mathbf{M}^n, (g_t)_{t \in [-1, 1]})$ be a Ricci flow on a compact, n -dimensional manifold \mathbf{M} with the property that $\nu[g_{-1}, 4] \geq -A$. Assume that $|R| \leq R_0$ on $\mathbf{M} \times [-1, 1]$ for some constant $0 \leq R_0 \leq 1$. Let $[t_1, t_2] \subset [0, 1]$ be a sub-interval with $t_2 - t_1 \leq \theta_6(1 - t_1)$ and consider two points $x_1, x_2 \in \mathbf{M}$. Then

$$d_{t_2}(x_1, x_2) \leq \exp(C_6 R_0^{1/2} \sqrt{t_2 - t_1}) d_{t_1}(x_1, x_2) + C_6 \sqrt{t_2 - t_1}.$$

Proof. Set $d := d_{t_2}(x_1, x_2)$. For $i = 1, 2$ let $u_i \in C^0(\mathbf{M} \times [t_1, t_2]) \cap C^\infty(\mathbf{M} \times [t_1, t_2])$ be a solution to the backwards (not the conjugate!) heat equation

$$(4.3) \quad -\partial_t u_i = \Delta u_i, \quad u_i(\cdot, t_2) = d_{t_2}(x_i, \cdot)$$

and let $v_i \in C^0(\mathbf{M} \times [t_1, t_2]) \cap C^\infty(\mathbf{M} \times [t_1, t_2])$ be a solution to the conjugate heat equation

$$-\partial_t v_i = \Delta v_i - Rv_i, \quad v_i(\cdot, t_2) = d_{t_2}(x_i, \cdot).$$

Note that by the maximum principle, we have on $\mathbf{M} \times [t_1, t_2]$

$$(4.4) \quad u_1 + u_2 \geq \min_{\mathbf{M}} (u_1(\cdot, t_2) + u_2(\cdot, t_2)) \geq \min_{\mathbf{M}} (d_{t_2}(x_1, \cdot) + d_{t_2}(x_2, \cdot)) \geq d.$$

We also claim that we have for all $t \in [t_1, t_2]$

$$(4.5) \quad u_i(\cdot, t) \leq e^{R_0(t_2 - t)} v_i(\cdot, t).$$

This inequality follows by the maximum principle and by the fact that whenever $v_i \geq 0$, we have

$$(-\partial_t - \Delta)(e^{R_0(t_2 - t)} v_i(\cdot, t)) = e^{R_0(t_2 - t)} R_0 v_i(\cdot, t) - e^{R_0(t_2 - t)} R(\cdot, t) v_i(\cdot, t) \geq 0.$$

We now make use of the fact that for any $x \in \mathbf{M}$,

$$v_i(x, t_1) = \int_{\mathbf{M}} K(y, t_2; x, t_1) v_i(y, t_2) dg_{t_2}(y) = \int_{\mathbf{M}} K(y, t_2; x, t_1) d_{t_2}(x_i, y) dg_{t_2}(y)$$

and

$$K(y, t_2; x, t_1) < \frac{C_1^*}{(t_2 - t_1)^{n/2}} \exp\left(-\frac{d_{t_2}^2(x, y)}{C_2^*(t_2 - t_1)}\right),$$

for some constants C_1^*, C_2^* , which depend only on A . Note that the latter inequality is similar to (3.2) except that the distance between x, y is taken at time t_2 . This inequality follows from [BZ15, Theorem 1.4] and the subsequent comment in that paper. We can hence estimate, similarly as in the proof of Lemma 4.1,

$$v_i(x_i, t_1) \leq \int_{\mathbf{M}} \frac{C_1^*}{(t_2 - t_1)^{n/2}} \exp\left(-\frac{d_{t_2}^2(x_i, y)}{C_2^*(t_2 - t_1)}\right) d_{t_2}(x_i, y) dg_{t_2}(y) \leq C \sqrt{t_2 - t_1}.$$

So, using (4.5), we have

$$u_i(x_i, t_1) \leq C e^{R_0(t_2 - t_1)} \sqrt{t_2 - t_1} \leq 10C \sqrt{t_2 - t_1}.$$

So by (4.4) we have

$$u_1(x_2, t_1) \geq d - u_2(x_2, t_1) \geq d - 10C \sqrt{t_2 - t_1}.$$

This implies

$$(4.6) \quad |u_1(x_1, t_1) - u_1(x_2, t_2)| \geq d - 20C \sqrt{t_2 - t_1}.$$

Taking derivatives of (4.3), we obtain the evolution inequality

$$-\partial_t |\nabla u_1| \leq \Delta |\nabla u_1| + 4|\text{Ric}| \cdot |\nabla u_1| \leq \Delta |\nabla u_1| + (4 + \sqrt{n})|\text{Ric}| \cdot |\nabla u_1| - R|\nabla u_1|,$$

which holds in the barrier sense. Note that by definition $|\nabla u_1(\cdot, t_2)| \leq 1$. So, by Proposition 3.5, we have for sufficiently small θ_6

$$|\nabla u_1(\cdot, t_1)| \leq 1 + CR_0^{1/2} \sqrt{t_2 - t_1}.$$

So, using (4.6), we obtain

$$\begin{aligned} d_{t_2}(x_1, x_2) - 10C\sqrt{t_2 - t_1} &\leq |u(x_1, t_1) - u(x_2, t_2)| \\ &\leq (1 + CR_0^{1/2} \sqrt{t_2 - t_1})d_{t_1}(x_1, x_2) \leq \exp(CR_0^{1/2} \sqrt{t_2 - t_1})d_{t_1}(x_1, x_2). \end{aligned}$$

This finishes the proof. \square

Next, we remove the assumption $t_2 - t_1 \leq \theta_6(1 - t_1)$ from Lemma 4.2.

Lemma 4.3. *For every $A < \infty$ there is a constant $C_7 = C_7(A) < \infty$ such that the following holds:*

Let $(\mathbf{M}^n, (g_t)_{t \in [-1, 1]})$ be a Ricci flow on a compact, n -dimensional manifold \mathbf{M} with the property that $\nu[g_{-1}, 4] \geq -A$. Assume that $|R| \leq R_0$ on $\mathbf{M} \times [-1, 1]$ for some constant $0 \leq R_0 \leq 1$. Let $0 \leq t_1 \leq t_2 \leq 1$ and consider two points $x, y \in \mathbf{M}$. Then

$$d_{t_2}(x, y) \leq \exp(C_7 R_0^{1/2} \sqrt{t_2 - t_1})d_{t_1}(x, y) + C_7 \sqrt{t_2 - t_1}.$$

Proof. In the case in which $t_2 - t_1 \leq \theta_6(1 - t_1)$, the bound follows immediately from Lemma 4.2. Let us now assume that $t_2 - t_1 > \theta_6(1 - t_1)$. By continuity we may also assume without loss of generality that $t_2 < 1$.

Choose times

$$t'_k := 1 - (1 - \theta_6)^k(1 - t_1)$$

and observe that $t'_0 = t_1$ and

$$t'_{k+1} - t'_k = \theta_6(1 - \theta_6)^k(1 - t_1) = \theta_6(1 - t'_k).$$

So by Lemma 4.2

$$\begin{aligned} d_{t'_k}(x, y) &\leq \exp\left(C_6 R_0^{1/2} \sum_{l=1}^k \sqrt{t'_l - t'_{l-1}}\right)d_{t_1}(x, y) \\ &\quad + C_6 \sum_{l=1}^k \exp\left(C_6 R_0^{1/2} \sum_{j=l+1}^k \sqrt{t'_j - t'_{j-1}}\right) \sqrt{t'_l - t'_{l-1}}. \end{aligned}$$

Since

$$\sum_{l=1}^k \sqrt{t'_l - t'_{l-1}} = \sum_{l=1}^k \sqrt{\theta_6(1 - \theta_6)^{l/2} \sqrt{1 - t_1}} \leq C' \sqrt{1 - t_1}$$

and

$$\begin{aligned} \sum_{l=1}^k \exp \left(C_6 R_0^{1/2} \sum_{j=l+1}^k \sqrt{t'_j - t'_{j-1}} \right) \sqrt{t'_l - t'_{l-1}} \\ \leq \sum_{l=1}^k \exp \left(C_6 C' R_0^{1/2} \sqrt{1-t_1} \right) \sqrt{t'_l - t'_{l-1}} \leq C'' \sqrt{1-t_1}, \end{aligned}$$

we find that for a generic constant $C < \infty$

$$d_{t'_k}(x, y) \leq \exp(CR_0^{1/2}\sqrt{1-t_1})d_{t_1}(x, y) + C\sqrt{1-t_1}.$$

Choose now k such that $t'_k \leq t_2 < t'_{k+1}$. Then $t_2 - t'_k \leq t'_{k+1} - t'_k \leq \theta_6(1-t'_1)$, so again by Lemma 4.2, we have

$$\begin{aligned} d_{t_2}(x, y) &\leq \exp(C_6 R_0^{1/2} \sqrt{t_2 - t'_k}) d_{t'_k}(x, y) + C_6 \sqrt{t_2 - t'_k} \\ &\leq \exp((C+C_6)R_0^{1/2}\sqrt{1-t_1})d_{t_1}(x, y) + C \exp(1+C_6)\sqrt{1-t_1} + C_6 \sqrt{1-t_1}. \end{aligned}$$

The claim now follows using $\sqrt{1-t_1} < \theta_6^{-1/2}\sqrt{t_2 - t_1}$. \square

We can finally prove Theorem 1.1.

Proof of Theorem 1.1. Consider the Ricci flow $(\mathbf{M}^n, (g_t)_{t \in [0,1]})$ with $\nu[g_0, 1+A^{-1}] \geq -A$ and $|R| \leq R_0$ for $0 \leq R_0 \leq A$. After replacing A by $4A+2$, we may assume without loss of generality that $A > 2$ and that we even have $\nu[g_0, 1+4A^{-1}] \geq -A$.

We will first prove the distance bounds for the case in which $t_1 > 0$ and $t_2 \leq (1+A^{-1})t_1$. By monotonicity of ν (compare with [BZ15, sec 2]), we find that for any $t \in [0, 1]$ we have

$$\nu[g_t, 4A^{-1}] \geq \nu[g_0, 1+4A^{-1}] \geq -A.$$

Restrict the flow to the time-interval $[(1-A^{-1})t_1, (1+A^{-1})t_1]$ and parabolically rescale by $A^{1/2}t_1^{-1/2}$ to obtain a flow $(\tilde{g}_t)_{t \in [A-1, A+1]}$. Then $\nu[\tilde{g}_{A-1}, 4] \geq -A$ and $|\tilde{R}| \leq \tilde{R}_0 := A^{-1}t_1 R_0 \leq 1$. Then t_1, t_2 correspond to times $\tilde{t}_1 := A, \tilde{t}_2 := At_1^{-1}t_2$ and we have

$$\tilde{R}_0^{1/2} \sqrt{\tilde{t}_2 - \tilde{t}_1} = R_0^{1/2} \sqrt{t_2 - t_1}.$$

So the distance bounds follow from Lemmas 4.1 and 4.3.

Consider now the case in which $t_2 > (1+A^{-1})t_1$. So $t_1 < \lambda t_2$, where $\lambda := (1+A^{-1})^{-1} < 1$. By continuity we may assume without loss of generality that $t_1 > 0$. Then we can find $1 \leq k_2 < k_1$ such that $t_1 \in [\lambda^{k_1}, \lambda^{k_1-1}]$ and $t_2 \in [\lambda^{k_2}, \lambda^{k_2-1}]$. Using our previous conclusions, we find

$$d_{t_2}(x, y) \geq d_{\lambda^{k_2}}(x, y) - C\sqrt{\lambda^{k_2}} \geq \dots \geq d_{t_1}(x, y) - C \sum_{l=k_1}^{k_2} \sqrt{\lambda^l} \geq d_{t_1}(x, y) - C'C\lambda^{k_2/2}.$$

Since $t_1 < \lambda t_2$, we have $\sqrt{t_2 - t_1} > \sqrt{(1-\lambda)t_2} > \sqrt{1-\lambda}\sqrt{\lambda^{k_2}}$. So

$$d_{t_2}(x, y) \geq d_{t_1}(x, y) - C'C(1-\lambda)^{-1/2}\sqrt{t_2 - t_1}.$$

This establishes the lower bound.

For the upper bound, set $t'_0 := t_1$, $t'_1 := \lambda^{k_1-1}$, \dots , $t'_{k_1-k_2} := \lambda^{k_2}$, $t'_{k_1-k_2+1} := t_2$. Then we have by our previous conclusions

$$\begin{aligned} d_{t_2}(x, y) &\leq \exp \left(CR_0^{1/2} \sum_{l=1}^{k_1-k_2+1} \sqrt{t'_l - t'_{l-1}} \right) d_{t_1}(x, y) \\ &\quad + C \sum_{l=1}^{k_2-k_1+1} \exp \left(CR_0^{1/2} \sum_{j=l+1}^{k_1-k_2+1} \sqrt{t'_j - t'_{j-1}} \right) \sqrt{t'_l - t'_{l-1}} \end{aligned}$$

Similarly as in the proof of Lemma 4.3, we conclude

$$d_{t_2}(x, y) \leq \exp \left(CR_0^{1/2} \sqrt{\lambda^{k_2}} \right) d_{t_1}(x, y) + C \sqrt{\lambda^{k_2}}.$$

Again, using $\sqrt{t_2 - t_1} > \sqrt{1 - \lambda} \sqrt{\lambda^{k_2}}$, we get the desired bound. \square

5. PROOF OF COROLLARY 1.3

Proof of Corollary 1.3. For each i consider the metric \bar{d}^i on \mathbf{M}^i with

$$\bar{d}^i(x, y) := \int_0^1 d_t^i(x, y) dt.$$

Note that by the Hölder bound in Theorem 1.1 there is a uniform constant $c' > 0$ such that for all $t, t' \in [0, 1]$ we have $d_{t'}^i(x, y) > \frac{1}{2} d_t^i(x, y)$ whenever $|t - t'| \leq c'(d_t^i(x, y))^2$. So there is a uniform constant $c > 0$ such that for all $t \in [0, 1]$

$$(5.1) \quad \bar{d}^i(x, y) \geq c \left(\min \{d_t^i(x, y), 1\} \right)^3.$$

So by the triangle inequality and Theorem 1.1, for any $A < \infty$ there is a constant $C < \infty$ such that for any $x, y, x', y' \in \mathbf{M}$ and $t, t' \in [0, 1]$ with $\bar{d}^i(x, y) + \bar{d}^i(x, x') + \bar{d}^i(y, y') < A$ we have

$$(5.2) \quad |d_t^i(x, y) - d_{t'}^i(x', y')| \leq C(\bar{d}^i(x, x'))^{1/3} + C(\bar{d}^i(y, y'))^{1/3} + C|t - t'|^{1/2}.$$

We first argue that the sequence $(\mathbf{M}^i, \bar{d}^i)$ is uniformly totally bounded in the following sense: For any $0 < a < b$ there is a number $N = N(a, b) < \infty$ such that for any i and any $x \in \mathbf{M}^i$, the ball $\overline{B}^i(x, b) := \{z \in \mathbf{M}^i : \bar{d}^i(x, z) < b\}$ contains at most N pairwise disjoint balls $\overline{B}^i(y_j, a)$, $j = 1, \dots, m$. Fix $0 < a < b$ and assume without loss of generality that $a < 1$. By (5.1) there is a constant $b' = b'(b) < \infty$ such that $\overline{B}^i(x, b) \subset B^i(x, t, b')$ for all $t \in [0, 1]$.

Assume that $y_1, \dots, y_m \in \overline{B}^i(x, b)$ such that the balls $\overline{B}^i(y_j, a)$ are pairwise disjoint. This implies $\bar{d}^i(y_{j_1}, y_{j_2}) \geq 2a$ for all $j_1 \neq j_2$. By the Hölder bound in Theorem 1.1, we may find a large integer $L = L(a) < \infty$ such that whenever $\bar{d}^i(y, y') \geq 2a$ for some points $y, y' \in \mathbf{M}^i$, then $d_{\frac{l}{L}}^i(y, y') > a$ for some $l \in \{1, \dots, L\}$. So for any $j_1 \neq j_2$, there is an $l_{j_1, j_2} \in \{1, \dots, L\}$ such that

$$d_{\frac{l_{j_1, j_2}}{L}}^i(y_{j_1}, y_{j_2}) > a.$$

This implies the following statement: If we form the L -fold Cartesian product $\mathbf{M}^{i,L} := (\mathbf{M}^i)^L = \mathbf{M} \times \dots \times \mathbf{M}$ equipped with the metric $g_{\frac{1}{L}}^i \oplus \dots \oplus g_{\frac{L-1}{L}}^i$ and if we define $y_j^L :=$

$(y_j, \dots, y_j) \in \mathbf{M}^{i,L}$, then $d^{\mathbf{M}^{i,L}}(y_{j_1}^L, y_{j_2}^L) > a$ for any $j_1 \neq j_2$. So the $\frac{1}{2}a$ -balls around $y_{j_1}^L$ are pairwise disjoint and contained in $B^i(x, \frac{1}{L}, b' + a) \times \dots \times B^i(x, \frac{L-1}{L}, b' + a)$. Using (2.1) and Lemma 2.1, we conclude that

$$\left(c\left(\frac{a}{\sqrt{L}}\right)^n\right)^L \cdot m \leq \left(C_0(b')^n e^{C_0 b'}\right)^L,$$

which yields an upper bound on m . So the sequence $(\mathbf{M}^i, \bar{d}^i)$ is in fact uniformly totally bounded.

We may now pass to a subsequence and assume that $(\mathbf{M}^i, \bar{d}^i, x_i)$ converges to some metric space $(\overline{\mathbf{M}}, \bar{d}, \bar{x})$ in the pointed Gromov-Hausdorff sense. By (5.2) and Arzelá-Ascoli and after passing to another subsequence, the sequence of time-dependent metrics $(d^i)_{t \in [0,1]}$ converges locally uniformly to a time-dependent, continuous family of pseudo-metrics $(d_t^\infty)_{t \in [0,1]}$ on $\overline{\mathbf{M}}$. So for any $t \in [0, 1]$, the pointed metric spaces $(\mathbf{M}^i, d_t^i, x_i)$ converge to $(\overline{\mathbf{M}}/\sim_t, d_t^\infty, \bar{x})$ in the pointed Gromov-Hausdorff sense. Passing to another subsequence once again, and using (2.1), we can ensure that also the volume forms dg_t^i converge uniformly for every rational $t \in [0, 1]$. Since $e^{-A|t_2-t_1|} dg_{t_1}^i \leq dg_{t_2}^i \leq e^{A|t_2-t_1|} dg_{t_1}^i$, the convergence holds for any $t \in [0, 1]$. The doubling property for balls of radius less than \sqrt{t} follows from (2.1) after parabolic rescaling by $(\frac{1}{2}t)^{-1/2}$. \square

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